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On a Generalized R^h − Birecurrent Finsler Space

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Abstract: **In the present paper, a Finsler space whose curvature tensor R^i_{jkh} satisfies $R^i_{jkh|\ell|m} = a_{\ell m}R^i_{jkh} + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $R^i_{jkh} \neq 0$, where $a_{\ell m}$ and $b_{\ell m}$ are non-zero covariant tensor fields of second order called recurrence tensor fields, is introduced, such space is called as a generalized R^h −birecurrent Finsler space . The associate tensor R_{jrkh} of Cartan's third curvature tensor R^i_{jkh} , the torsion tensor H^i_{kh} ,the deviation tensor R^i_h, the Ricci tensor R_{jk}, the vector H_k and the scalar curvature R of such space are non-vanishing. Under certain conditions, a generalized R^h −birecurrent Finsler space becomes Landsberg space . Some conditions have been pointed out which reduce a generalized R^h −birecurrent Finsler space $F_n(n > 2)$ into Finsler space of scalar curvature.**

Keywords: **Finsler space; Generalized R^h −birecurrent Finsler space; Ricci tensor; Landsberg space; Finsler space of scalar curvature.**

1. INTRODUCTION

H.S. Ruse [4] considered a three dimensional Riemannian space having the recurrent of curvature tensor and he called such space as *Riemannian space of recurrent curvature*. This idea was extended to n-dimensional Riemannian and non-Riemannian space by A.G. Walker [1],Y.C. Worg [9] ,Y.C. Worg and K. Yano [10] and others .

This idea was extended to Finsler spaces by A.Moor [2] for the first time . Due to different connections of Finsler space, the recurrent of Cartan`s third curvature tensor $R^i_{jkh}$ have been discussed by, R.Verma [7] , birecurrent of Cartan`s third curvature tensor R^i_{jkh} have been discussed by S.Dikshit [8] and the generalized birecurrent of Cartan`s third curvature tensor R^i_{jkh} have been discussed by F.Y.A.Qasem [3] .P.N.Pandey, S.Saxena and A.Goswami [6] interduced a generalized H-recurrent Finsler space.

Let F_n be An n-dimensional Finsler space equipped with the metric function a $F(x,y)$ satisfying the request conditions [4] .

The vectors y_i , $y^i$and the metric tensor g_{ij} satisfies the following relations

(1.1) a) $y_i\, y^i = F^2$ b) $g_{ij} = \dot\partial_i y_j = \dot\partial_j y_i$ c) $y_{i|k} = 0$

d) $y^i_{\ |k} = 0$ e) $g_{ij|k} = 0$ f) $g^{ij}_{|k} = 0$.

Thus the unit vector l^i and the associate vector l_i is defined by

(1.2) a) $l^i = \frac{y^i}{F}$ b) $l_i = g_{ij}l^j = \dot\partial_i F = \frac{y_i}{F}$.

The two processes of covariant differentiation, defined above commute with the partial

(1.3) a) $\dot\partial_j(X^i_{|k}) - (\dot\partial_j X^i)_{|k} = X^r(\dot\partial_j \Gamma^{*i}_{rk}) - (\dot\partial_r X^i) P^r_{jk}$,

b) $P^r_{jk} = (\dot\partial_j \Gamma^{*r}_{hk}) y^h = \Gamma^{*r}_{jhk}\ y^h$,

c) $\Gamma^{*i}_{jkh}\, y^h = G^i_{jkh}\, y^h = 0$,

d) $P^i_{jk} y^j = 0$,

e) $g_{ir}\, P^i_{kh} = P_{rkh}$.

The tensor H^i_{jkh} satisfies the relation

(1.4) $H^i_{jkh}\, y^j = H^i_{kh}$.

(1.5) $H^i_{jkh} = \dot\partial_j H^i_{kh}$.

The torsion tensor H^i_{kh} satisfies

(1.6) $H^i_{kh}\, y^h = H^i_k$,

(1.7) $R^i_{jkh}\, y^j = H^i_{kh}$,

(1.8) $H_{jk} = H^i_{jki}$,

(1.9) $H_k = H^i_{ki}$,

and

(1.10) $H = \frac{1}{n-1} H^i_i$.

where H_{jk} and H are called *h-Ricci tensor* [5] and *curvature scalar* respectively. Since contraction of the indices does not affect the homogeneity in y^i, hence the tensors H_{rk} , H_r and the scalar H are also homogeneous of degree zero, one and two in y^i respectively . The above tensors are also connected by

(1.11) $H_{jk}\, y^j = H_k$,

(1.12) $H_{jk} = \dot\partial_j H_k$,

(1.13) $H_k\, y^k = (n-1) H$.

The tensors H^i_h , H^i_{kh} and H^i_{jkh} also satisfy the following :

(1.14) $H^i_{kh} = \dot\partial_k H^i_h$,

(1.15) $g_{ij}\, H^i_k = g_{ik}\, H^i_j$.

The associate tensor R_{ijkh} of Catan`s third curvature tensor R^i_{jkh} is given by

(1.16) $R_{ijkh} = g_{rj}\, R^r_{ikh}$.

The necessary and sufficient condition for a Finsler space $F_n (n > 2)$ to be a Finsler space of scalar curvature is given by

(1.17) $H^i_h = F^2 R(\delta^i_h - l^i\, l_h)$.

A Finsler space F_n is said to be Landsberg space if satisfies

(1.18) $y_r G^r_{jkh} = -2C_{jkh|m} y^m = -2P_{jkh} = 0$.

The Ricci tensor R_{jk} of the curvature tensor R^i_{jkh} , the tensor R^r_h and the scalar R are given by

(1.19) a) $R^i_{jki} = R_{jk}$,

b) $R^r_{ikh} g^{ik} = R^r_h$,

c) $g^{jk} R_{jk} = R$.

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2. GENERALIZED R^h –BIRECURRENT FINSLER SPACE

Let us consider a Finsler space F_n whose Cartan's third curvature tensor R^i_{jkh} satisfies

(2.1) $R^i_{jkh|\ell} = \lambda_\ell R^i_{jkh} + \mu_\ell(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $R^i_{jkh} \neq 0$, where λ_ℓ and μ_ℓ are non-zero covariant vector fields and called the recurrence vector fields. Such space called it as a generalized R^h- recurrent Finsler space.

Differentiating (2.1) covariantly with respect to x^m in the sense of Cartan and using (1.1.e), we get

(2.2) $$R^i_{jkh|\ell|m} = \lambda_{\ell|m} R^i_{jkh} + \lambda_\ell H^i_{jkh|m} + \mu_{\ell|m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

Using (2.1) in (2.2) we get

$$R^i_{jkh|\ell|m} = (\lambda_{\ell|m} + \lambda_\ell \lambda_m) R^i_{jkh} + (\lambda_\ell \mu_m + \mu_{\ell|m})(\delta^i_k g_{jh} - \delta^i_h g_{jk}),$$

which can be written as

(2.3) $$R^i_{jkh|\ell|m} = a_{\ell m} R^i_{jkh} + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}),\ R^i_{jkh} \neq 0\ ,$$

Where $a_{\ell m} = \lambda_{\ell|m} + \lambda_\ell \lambda_m$ and $b_{\ell m} = \lambda_\ell \mu_m + \mu_{\ell|m}$ are non-zero covariant tensor fields of second order and called recurrence tensor fields.

Definition 2.1. If Cartan's third curvature tensor R^i_{jkh} of a Finsler space satisfying the condition (2.3), where $a_{\ell m}$ and $b_{\ell m}$ are non-zero covariant tensor fields of second order , the space and the tensor will be called generalized $R^h -$ birecurrent Finsler space, we shall denote such space briefly by $GR^h - BR - F_n$.

However, if we start from condition (2.3), we cannot obtain the condition (2.1), we may conclude

Theorem 2.1. Every generalized $R^h -$ recurrent Finsler space is generalized $R^h -$ birecurrent Finsler space**,** but the converse need not be true.

Transvecting (2.3) by the metric tensor g_{ir} , using (1.1e) and (1.16) ,we get

(2.4) $$R_{jrkh|\ell|m} = a_{\ell m} R_{jrkh} + b_{\ell m}(g_{kr} g_{jh} - g_{hr} g_{jk}).$$

Transvecting (2.3) by y^j , using (1.1d) and (1.7) we get

(2.5) $$H^i_{kh|\ell|m} = a_{\ell m} H^i_{kh} + b_{\ell m}(\delta^i_k y_h - \delta^i_h y_k).$$

Further transvecting (2.5) by y^k , using (1.1d) and (1.6), we get

(2.6) $$H^i_{h|\ell|m} = a_{\ell m} H^i_h + b_{\ell m}(y^i y_h - \delta^i_h F^2)$$

Thus we have

Theorem 2.2. In $GR^h - BR - F_n$,the associate tensor R_{jrkh} of Cartan's third curvature tensor R^i_{jkh} , the torsion tensor H^i_{kh} and the deviation tensor H^i_h are non- vanishing.

Contracting the indices i and h in equations (2.3), (2.5) and (2.6) , using (1.19a), (1.9), (1.10) and (1.1 a), we get

(2.7) $$R_{jk|\ell|m} = a_{\ell m} R_{jk} + (1-n) b_{\ell m} g_{jk}\ .$$

(2.8) $$H_{k|\ell|m} = a_{\ell m} H_k + (1-n) b_{\ell m} y_k\ .$$

(2.9) $$H_{|\ell|m} = a_{\ell m} H - b_{\ell m} F^2\ .$$

Transvecting (2.3) and (2.7) by g^{jk} , using (1.1f) , (1.19b) and (1.19c), we get

(2.10) $$R^i_{h|\ell|m} = a_{\ell m} R^i_h + b_{\ell m}(y^i y_h - \delta^i_h)\ .$$

(2.11) $$R_{|\ell|m} = a_{\ell m} R + (1-n)\, b_{\ell m}\ .$$

Thus, we conclude

Theorem 2.3. In $GR^h - BR - F_n$,the Ricci tensor R_{jk}, the curvature vector H_k ,the scalar curvature H the deviation tensor R^i_h and the scalar curvature tensor R are non- vanishing.

Differentiating (2.5) partially with respect to y^j , using (1.5) and (1.1b), we get

(2.12) $$\dot\partial_j\left(H^i_{kh|\ell|m}\right) = (\dot\partial_j a_{\ell m})H^i_{kh} + a_{\ell m}H^i_{jkh} + (\dot\partial_j b_{\ell m})(\delta^i_k y_h - \delta^i_h y_k) + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

Using commutation formula exhibited by (1.3b) for $(H^i_{kh|\ell})$ in (2.12), we get

(2.13) $$\left\{\dot\partial_j\left(H^i_{kh|\ell}\right)\right\}_{|m} + H^r_{kh|\ell}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\dot\partial_j \Gamma^{*r}_{km}) - H^i_{rk|\ell}(\dot\partial_j \Gamma^{*r}_{hm}) - H^r_{kh|r}(\dot\partial_j \Gamma^{*i}_{m\ell}) - \dot\partial_r\left(H^i_{kh|\ell}\right)P^r_{jm} = (\dot\partial_j a_{\ell m})H^i_{kh} + a_{\ell m}H^i_{jkh} + (\dot\partial_j b_{\ell m})(\delta^i_k y_h - \delta^i_h y_k) + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

Again applying the commutation formula exhibited by (1.3a) for (H^i_{kh}) in (2.13) and using (1.5), we get

(2.14) $$\left\{H^i_{jkh|\ell} + H^r_{kh}(\dot\partial_j \Gamma^{*i}_{r\ell}) - H^i_{rh}(\dot\partial_j \Gamma^{*r}_{k\ell}) - H^i_{rk}(\dot\partial_j \Gamma^{*r}_{h\ell}) - H^i_{rkh}P^r_{j\ell}\right\}_{|m} + H^r_{kh|\ell}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\dot\partial_j \Gamma^{*r}_{km}) - H^i_{rk|\ell}(\dot\partial_j \Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{m\ell}) - \left\{H^i_{rkh|\ell} + H^s_{kh}(\dot\partial_r \Gamma^{*i}_{s\ell}) - H^i_{sh}(\dot\partial_r \Gamma^{*s}_{k\ell}) - H^i_{sk}(\dot\partial_r \Gamma^{*s}_{h\ell}) - H^i_{skh}P^s_{r\ell}\right\}P^r_{jm} = (\dot\partial_j a_{\ell m})H^i_{kh} + a_{\ell m}H^i_{jkh} + (\dot\partial_j b_{\ell m})(\delta^i_k y_h - \delta^i_h y_k) + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

This shows that

(2.15) $$H^i_{jkh|\ell|m} = a_{\ell m}H^i_{jkh} + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

if and only if

(2.16) $$\left\{H^r_{kh}(\dot\partial_j \Gamma^{*i}_{r\ell}) - H^i_{rh}(\dot\partial_j \Gamma^{*r}_{k\ell}) - H^i_{rk}(\dot\partial_j \Gamma^{*r}_{h\ell}) - H^i_{rkh}P^r_{j\ell}\right\}_{|m} + H^r_{kh|\ell}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\dot\partial_j \Gamma^{*r}_{km}) - H^i_{rk|\ell}(\dot\partial_j \Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{m\ell}) - \left\{H^i_{rkh|\ell} + H^s_{kh}(\dot\partial_r \Gamma^{*i}_{s\ell}) - H^i_{sh}(\dot\partial_r \Gamma^{*s}_{k\ell}) - H^i_{sk}(\dot\partial_r \Gamma^{*s}_{h\ell}) - H^i_{skh}P^s_{r\ell}\right\}P^r_{jm} = (\dot\partial_j a_{\ell m})H^i_{kh} + (\dot\partial_j b_{\ell m})(\delta^i_k y_h - \delta^i_h y_k).$$

Contracting the i and h in (2.14) and using (1.8), we get

(2.17) $$H_{jk|\ell|m} + \left\{H^r_{kp}(\dot\partial_j \Gamma^{*p}_{r\ell}) - H_r(\dot\partial_j \Gamma^{*r}_{k\ell}) - H^p_{rk}(\dot\partial_j \Gamma^{*r}_{p\ell}) - H_{rk}P^r_{j\ell}\right\}_{|m} + H^r_{kp|\ell}(\dot\partial_j \Gamma^{*p}_{rm}) - H_{r|\ell}(\dot\partial_j \Gamma^{*r}_{km}) - H^p_{rk|\ell}(\dot\partial_j \Gamma^{*r}_{pm}) - H_{k|r}(\dot\partial_j \Gamma^{*r}_{m\ell}) - \{H_{rk|\ell} + H^s_{kp}(\dot\partial_r \Gamma^{*p}_{s\ell}) - H_s(\dot\partial_r \Gamma^{*s}_{k\ell}) - H^p_{sk}(\dot\partial_r \Gamma^{*s}_{p\ell}) - H_{sk}P^s_{r\ell}\}P^r_{jm} = (\dot\partial_j a_{\ell m})H_k + a_{\ell m}H_{jk} + (1-n)(\dot\partial_j b_{\ell m})y_k + (1-n)d_{\ell m}g_{jk}.$$

This shows that

(2.18) $$H_{jk|\ell|m} = a_{\ell m}H_{jk} + (1-n)d_{\ell m}g_{jk}.$$

if and only if

(2.19) $$\left\{H^r_{kp}(\dot\partial_j \Gamma^{*p}_{r\ell}) - H_r(\dot\partial_j \Gamma^{*r}_{k\ell}) - H^p_{rk}(\dot\partial_j \Gamma^{*r}_{p\ell}) - H_{rk}P^r_{j\ell}\right\}_{|m} + H^r_{kp|\ell}(\dot\partial_j \Gamma^{*p}_{rm}) - H_{r|\ell}(\dot\partial_j \Gamma^{*r}_{km}) - H^p_{rk|\ell}(\dot\partial_j \Gamma^{*r}_{pm}) - H_{k|r}(\dot\partial_j \Gamma^{*r}_{m\ell}) - \{H_{rk|\ell} + H^s_{kp}(\dot\partial_r \Gamma^{*p}_{s\ell}) - H_s(\dot\partial_r \Gamma^{*s}_{k\ell}) - H^p_{sk}(\dot\partial_r \Gamma^{*s}_{p\ell}) - H_{sk}P^s_{r\ell}\}P^r_{jm}$$

$$= (\dot{\partial}_j a_{\ell m})H_k + (1-n)(\dot{\partial}_j b_{\ell m})y_k .$$

Thus , we have

Theorem2.4. In $GR^h - BR - F_n$, Berwald curvature tensor H^i_{jkh} and Ricci curvature tensor H_{jk} are non- vanishing if and only if conditions (2.16) and (2.19) hold, respectively.

Differentiating (2.8) partially with respect to y^j, using (1. 12) and (1.1b), we get

(2.20) $$\dot{\partial}_j(H_{k|\ell|m}) = (\dot{\partial}_j a_{\ell m})H_k + a_{\ell m}H_{jk} + (1-n)(\dot{\partial}_j b_{\ell m})y_k + (1-n)b_{\ell m}\, g_{jk} .$$

Using the commutation formula exhibited by (1. 3a) for $(H_{k|\ell})$ and using (1.12), we get

(2.21) $$(\dot{\partial}_j H_{k|\ell})_{|m} - H_{r|\ell}(\dot{\partial}_j \Gamma^{*r}_{km}) - H_{k|r}(\dot{\partial}_j \Gamma^{*r}_{\ell m}) - (\dot{\partial}_r H_{k|\ell})P^r_{jm} = (\dot{\partial}_j a_{\ell m})H_k + a_{\ell m}H_{jk} + (1-n)(\dot{\partial}_j b_{\ell m})y_k + (1-n)b_{\ell m}g_{jk} .$$

Again using commutation formula exhibited by (1.3a) for (H_k) in (2.21) , we get

(2.22) $$\left\{(\dot{\partial}_j H_k)_{|\ell} - H_r(\dot{\partial}_j \Gamma^{*r}_{\ell k}) - (\dot{\partial}_r H_k)P^r_{j\ell}\right\}_{|m} - H_{r|\ell}(\dot{\partial}_j \Gamma^{*r}_{km}) - H_{k|r}(\dot{\partial}_j \Gamma^{*r}_{\ell m}) - \{(\dot{\partial}_r H_k)_{|\ell} - H_s(\dot{\partial}_r \Gamma^{*s}_{\ell k}) - (\dot{\partial}_s H_k)P^s_{r\ell}\}P^r_{jm} = (\dot{\partial}_j a_{\ell m})H_k + a_{\ell m}H_{jk} + (1-n)(\dot{\partial}_j b_{\ell m})y_k + (1-n)b_{\ell m}g_{jk} .$$

Using (1.12) and (2.18) in (2.22), we get

(2.23) $$\left\{-H_r(\dot{\partial}_j \Gamma^{*r}_{\ell k}) - (H_{kr})P^r_{j\ell}\right\}_{|m} - H_{r|\ell}(\dot{\partial}_j \Gamma^{*r}_{km}) - H_{k|r}(\dot{\partial}_j \Gamma^{*r}_{\ell m}) - \left\{H_{kr_{|\ell}} - H_s(\dot{\partial}_r \Gamma^{*s}_{\ell k}) - H_{ks}P^s_{r\ell}\right\}P^r_{jm} = (\dot{\partial}_j a_{\ell m})H_k + (1-n)(\dot{\partial}_j b_{\ell m})y_k .$$

Transvecting (2.23) by y^k , using (1.1d), (1.13) , (1.3b) and (1.1a), we get

$$-2H_{r|\ell}P^r_{jm} - (n-1)H_{|r}(\dot{\partial}_j \Gamma^{*r}_{\ell m}) = (n-1)(\dot{\partial}_j a_{\ell m})H - (n-1)(\dot{\partial}_j b_{\ell m})F^2.$$

Which can be written as

(2.24) $$(\dot{\partial}_j b_{\ell m}) = \frac{(\dot{\partial}_j a_{\ell m})H}{F^2} .$$

if and only if

(2.25) $$2H_{r|\ell}P^r_{jm} + (n-1)H_{|r}(\dot{\partial}_j \Gamma^{*r}_{\ell m}) = 0 .$$

If the tensor $a_{\ell m}$ is independent of y^i, the equation (2.24) shows that the tensor $b_{\ell m}$ is also independent of y^i. Conversely , if the tensor $b_{\ell m}$ is independent of y^i, we get $H\dot{\partial}_j a_{\ell m} = 0$. In view of theorem2.3, the condition $H\dot{\partial}_j a_{\ell m} = 0$ implies $\dot{\partial}_j a_{\ell m} = 0$, ,i.e. the covariant tensor $a_{\ell m}$ is also independent of y^i. This leads to

Theorem 2.5. The covariant tensor $b_{\ell m}$ is independent of the directional arguments if the covariant tensor $a_{\ell m}$ is independent of directional arguments if and only if conditions (2.25) and (2.19) hold.

Suppose the tensor $a_{\ell m}$ is not independent of y^i**,** then (2.23) and (2.24) together imply

(2.26) $$\left\{-H_r(\dot{\partial}_j \Gamma^{*r}_{\ell k}) - (H_{kr})P^r_{j\ell}\right\}_{|m} - H_{r|\ell}(\dot{\partial}_j \Gamma^{*r}_{km}) - H_{k|r}(\dot{\partial}_j \Gamma^{*r}_{\ell m}) - \{H_{kr|\ell} - H_s(\dot{\partial}_r \Gamma^{*s}_{\ell k}) - H_{ks}P^s_{r\ell}\}P^r_{jm} = (\dot{\partial}_j a_{\ell m})(H_k - \frac{(n-1)}{F^2}Hy_k) .$$

Transvecting (2.26) by y^m and using (1.1d) , (1.3c) and (1.3d), we get

(2.27) $\{-H_r(\dot{\partial}_j\Gamma^{*r}_{\ell k}) - (H_{kr})P^r_{j\ell}\}_{|m} y^m = (\dot{\partial}_j a_\ell - a_{j\ell})(H_k - \frac{(n-1)}{F^2} H y_k).$

where $a_{\ell m} y^m = a_\ell$

if

(2.28) $\{-H_r(\dot{\partial}_j\Gamma^{*r}_{\ell k}) - (H_{kr})P^r_{j\ell}\}_{|m} y^m = 0$, equation (2.27) implies at least one of the following conditions

(2.29) a) $a_{j\ell} = \dot{\partial}_j a_\ell,$ b) $H_k = \frac{(n-1)}{F^2} H y_k$

Thus, we have

Theorem 2.6. In $GR^h - BR - F_n$ for which the covariant tensor $a_{\ell m}$ is not independent of the directional arguments and if conditions (2.28) and (2.19) (2.25) hold, at least one of the conditions (2.29a) and (2.29b) hold.

Suppose (2.29b) holds equation (2.26) implies

(2.30) $$\left\{\frac{(n-1)}{F^2} H y_r \dot{\partial}_j \Gamma^{*r}_{\ell k} + H_{kr} P^r_{j\ell}\right\}_{|m} + \left\{\frac{(n-1)}{F^2} H y_r\right\}_{|\ell} \dot{\partial}_j \Gamma^{*r}_{km}$$
$$+ \left\{\frac{(n-1)}{F^2} H y_k\right\}_{|r} \dot{\partial}_j \Gamma^{*r}_{\ell m} + H_{kr|\ell} P^r_{jm} + \frac{(n-1)}{F^2} H y_s (\dot{\partial}_r \Gamma^{*s}_{\ell k}) P^r_{jm}$$
$$+ H_{ks} P^s_{r\ell} P^r_{jm} = 0 .$$

Transvecting (2.30) by y^j , using (1.1d), (1.3b) and (1.3d), we get

(2.31) $$\left\{\frac{(n-1)}{F^2} H y_r \mathrm{P}^r_{\ell k}\right\}_{|m} + \left\{\frac{(n-1)}{F^2} H y_r\right\}_{|\ell} P^r_{km} + \left\{\frac{(n-1)}{F^2} H y_k\right\}_{|r} \mathrm{P}^r_{\ell m} = 0 .$$

Thus, we have

Theorem 2.7. In $GR^h - BR - F_n$, we have the identity (2.31) provided (2.29b).

Transvecting (2.31) by the metric tensor g_{rj} , using (1.1e) and (1.3e), we get

(2.32) $$\left\{\frac{(n-1)}{F^2} H y_r P_{j\ell k}\right\}_{|m} + \left\{\frac{(n-1)}{F^2} H y_r\right\}_{|\ell} P_{jkm} + \left\{\frac{(n-1)}{F^2} H y_k\right\}_{|r} P_{j\ell m} = 0.$$

By using (1.1.c) , equation (1.22) can be written as

$$y_r (H P_{j\ell k})_{|m} + y_r H_{|\ell} P_{jkm} + y_k H_{|r} P_{j\ell m} = 0 .$$

In view of theorem2.3 , we have

(2. 33) $P_{j\ell m} = 0$.

if and only if

(2.34) $y_r (H P_{j\ell k})_{|m} + y_r H_{|\ell} P_{jkm} = 0$.

Therefore the space is Landsberg space.

Thus, we have

Theorem 2.8. An $GR^h - BR - F_n$ is Landsberg space if and only if conditions (2.34) and (2.29b) hold good.

If the covariant tensor $a_{j\ell} \neq \dot{\partial}_j a_\ell$, in view of theorem2.6, (2.29b) holds good. In view of this fact, we may rewrite theorem 2.8 in the following form

Theorem 2.9. An $GR^h - BR - F_n$ is necessarily Landsberg space if and only if conditions (2.34) and (2.29b) hold good and provided $a_{j\ell} \neq \dot{\partial}_j a_\ell$.

Using (2.15) in (2.14), we get

(2.35) $$\{H^r_{kh}(\dot{\partial}_j \Gamma^{*i}_{r\ell}) - H^i_{rh}(\dot{\partial}_j \Gamma^{*r}_{k\ell}) - H^i_{rk}(\dot{\partial}_j \Gamma^{*r}_{h\ell}) - H^i_{rkh} P^r_{j\ell}\}_{|m}$$
$$+ H^r_{kh|\ell}(\dot{\partial}_j \Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\dot{\partial}_j \Gamma^{*r}_{km}) - H^i_{rk|\ell}(\dot{\partial}_j \Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot{\partial}_j \Gamma^{*r}_{m\ell})$$

$$-\left\{H^i_{rkh|\ell} + H^s_{kh}(\dot{\partial}_r\Gamma^{*i}_{s\ell}) - H^i_{sh}(\dot{\partial}_r\Gamma^{*s}_{k\ell}) - H^i_{sk}(\dot{\partial}_r\Gamma^{*s}_{h\ell}) - H^i_{skh}P^s_{r\ell}\right\}P^r_{jm}$$

$$= (\dot{\partial}_j a_{\ell m})H^i_{kh} + (\dot{\partial}_j b_{\ell m})(\delta^i_k y_h - \delta^i_h y_k).$$

Transvecting (2.35) by y^k , using (1.1d), (1.1a), (1.3b), (1.4) and (1.6), we get

(2.36) $$\{H^r_h(\dot{\partial}_j\Gamma^{*i}_{r\ell}) - H^i_r(\dot{\partial}_j\Gamma^{*r}_{h\ell}) - 2H^i_{rh}P^r_{j\ell}\}_{|m} + H^r_{h|\ell}(\dot{\partial}_j\Gamma^{*i}_{rm}) - H^i_{rh|\ell}(P^r_{jm})$$

$$-H^i_{r|\ell}(\dot{\partial}_j\Gamma^{*r}_{hm}) - H^i_{h|r}(\dot{\partial}_j\Gamma^{*r}_{m\ell}) - \{H^i_{rh|\ell} + H^s_h(\dot{\partial}_r\Gamma^{*i}_{s\ell}) - H^i_s(\dot{\partial}_r\Gamma^{*s}_{h\ell})$$

$$-2H^i_{sh}P^s_{r\ell}\}P^r_{jm} = (\partial_j a_{\ell m})H^i_h + (\partial_j b_{\ell m})(y^i y_h - \delta^i_h F^2).$$

Substituting the value of $\dot{\partial}_j b_{\ell m}$ from (2. 24) , in (2. 36), we get

(2.37) $$\{H^r_h(\dot{\partial}_j\Gamma^{*i}_{r\ell}) - H^i_r(\dot{\partial}_j\Gamma^{*r}_{h\ell}) - 2H^i_{rh}P^r_{j\ell}\}_{|m} + H^r_{h|\ell}(\dot{\partial}_j\Gamma^{*i}_{rm}) - H^i_{rh|\ell}(P^r_{jm})$$

$$-H^i_{r|\ell}(\dot{\partial}_j\Gamma^{*r}_{hm}) - H^i_{h|r}(\dot{\partial}_j\Gamma^{*r}_{m\ell}) - \{H^i_{rh|\ell} + H^s_h(\dot{\partial}_r\Gamma^{*i}_{s\ell}) - H^i_s(\dot{\partial}_r\Gamma^{*s}_{h\ell})$$

$$-2H^i_{sh}P^s_{r\ell}\}P^r_{jm} = (\dot{\partial}_j a_{\ell m})[H^i_h - H(\delta^i_h - l^i l_h)].$$

if

(2.38) $$\{H^r_h(\dot{\partial}_j\Gamma^{*i}_{r\ell}) - H^i_r(\dot{\partial}_j\Gamma^{*r}_{h\ell}) - 2H^i_{rh}P^r_{j\ell}\}_{|m} + H^r_{h|\ell}(\dot{\partial}_j\Gamma^{*i}_{rm}) - H^i_{rh|\ell}(P^r_{jm})$$

$$-H^i_{r|\ell}(\dot{\partial}_j\Gamma^{*r}_{hm}) - H^i_{h|r}(\dot{\partial}_j\Gamma^{*r}_{m\ell}) - \{H^i_{rh|\ell} + H^s_h(\dot{\partial}_r\Gamma^{*i}_{s\ell}) - H^i_s(\dot{\partial}_r\Gamma^{*s}_{h\ell}) - 2H^i_{sh}P^s_{r\ell}\}P^r_{jm} = 0 .$$

We have at least one of the following conditions :

(2.39) a) $(\dot{\partial}_j a_{\ell m}) = 0,$ b) $H^i_h = H(\delta^i_h - l^i l_h).$

Putting $= F^2 R$, the equation (2. 39b) may be written as

(2.40) $$H^i_h = F^2 R(\delta^i_h - l^i l_h),$$

where $R \neq 0$. Therefore the space is a Finsler space of scalar curvature .

Thus , we have

Theorem 2.10. An $GR^h - BR - F_n$ for $n > 2$ admitting equation (2.38) holds is a Finsler space of scalar curvature provided $R \neq 0$, the covariant tensor $a_{\ell m}$ is not independent of directional arguments and condition (2.16) holds.

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